ON THE USE OF FRACTIONAL DIFFERENTIATION OPERATORS FOR THE DESCRIPTION OF ELASTIC-AFTEREFFECT PROPERTIES OF MATERIALS

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We report a study of the elastic-aftereffect properties of a medium whose shear deformations are described by the modified Voigt, Maxwell, and standard linear-body models in which the differentiation operators are replaced by fractional differentiation operators. The possibility of applying this type of description to various real materials is discussed.

Simple rheological models (Voigt, Maxwell, and standard linear-body models) are widely used for the qualitative description of the elastic-aftereffect properties of materials in the linear region. These models are simple and readily interpreted but have the important disadvantage that they do not take into account the spread of the relaxation spectra which usually occurs for real materials. The relaxation spectrum of a particular material can be taken into account in the simpler rheological models if we modify somewhat the viscous element in the rheological scheme. To achieve this it is sufficient to modify the Newtonian viscosity law

$$s_{ik} = 2\mu D e_{ik} \qquad \left(D \equiv \tau \frac{\partial}{\partial t}\right) \tag{1}$$

by introducing the replacement $D \to D^{\gamma}$, where s_{ik} and e_{ik} are the deviators of the stress deformation tensors, respectively, μ is the shear modulus, and τ is the relaxation time.

We shall restrict our attention to shear deformation, and write out Hooke's law for a viscoelastic medium in the form

$$s_{ik} = 2\mu(t) * e_{ik}, \qquad S_{ik} = 2ME_{ik}, \tag{2}$$

where the asterisk represents convolution in the segment (0, t) and S_{ik} denotes the one-sided Laplace transform of the function s_{ik}, and similarly for the other functions. The replacement of the operators D by the fractional differentiation operators means that we are transforming in the image space from the factors $p\tau$ to $(p\tau)^{\gamma}$, where $0 < \gamma \leq 1$. Bearing this in mind, we obtain the following expression for the elastic moduli M(p) for the Voigt, Maxwell, and standard linear-body models:

$$M = \mu_0 [1 + (p\tau_\sigma)^{\gamma}],$$

$$M = \frac{\mu_\infty}{1 + (p\tau_\varepsilon)^{-\gamma}}, \quad M = \mu_0 \frac{1 + (p\tau_\sigma)^{\gamma}}{1 + (p\tau_\varepsilon)^{\gamma}}. \tag{3}$$

Series-expanding the fractions for M and 1/M in powers of $(p\tau)^{-\gamma}$, and remembering that

$$\sum_{n=0}^{\infty} (-1)^n (p\tau)^{-\gamma(n+1)} \tag{4}$$

is the transform of

$$\frac{1}{\tau} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{\gamma(n+1)-1}}{\Gamma[\gamma(n+1)]} \equiv \text{Э}_{\gamma}(t, \tau), \tag{5}$$

we obtain the following results:

Voigt model

$$\mu(t) = \mu_0 [\delta(t) + \tau_\sigma^{\gamma} \delta^{(\gamma)}(t)], \qquad \frac{1}{\mu(t)} = \frac{1}{\mu_0} \text{Э}_{\gamma}(t, \tau_\sigma),$$

Maxwell model

$$\mu(t) = \mu_\infty [\delta(t) - \text{Э}_{\gamma}(t, \tau_\varepsilon)],$$

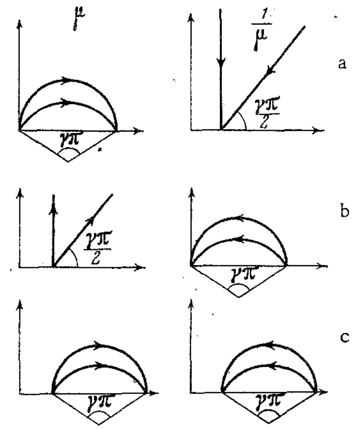
$$\frac{1}{\mu(t)} = \frac{1}{\mu_\infty} \left[\delta(t) + \frac{(t/\tau_\varepsilon)^{\gamma-1}}{\tau_\varepsilon \Gamma(\gamma)}\right],$$

standard linear-body model

$$\mu(t) = \mu_\infty \delta(t) - (\mu_\infty - \mu_0) \text{Э}_{\gamma}(t, \tau_\varepsilon)$$

$$\frac{1}{\mu(t)} = \frac{1}{\mu_\infty} \delta(t) + \left(\frac{1}{\mu_0} - \frac{1}{\mu_\infty}\right) \text{Э}_{\gamma}(t, \tau_\sigma), \tag{6}$$

where $\text{Э}_{\gamma}(t, \tau)$ is the fractional exponential function introduced by Rabotnov [1], $\delta(t)$ is the Dirac delta function, and $\delta^{(\gamma)}(t)$ is the derivative of this function of order γ. In the expressions given by (6) we have used the relation $\mu_\infty/\mu_0 = (\tau_\sigma/\tau_\varepsilon)^{\gamma}$. The Rabotnov operators have been used to solve a number of elastic-aftereffect problems for viscoelastic bodies [2, 3]. However, it is possible to use simpler operators to describe stress relaxation within the framework of the modified Voigt model. This is also the case for creep treated in terms of the modified Maxwell model.



We can readily obtain from Eq. (3) or (6) expression describing the reaction of the system to an instantaneous deformation $\dot{e}_{ik} = e_{ik}{}^{\circ}\delta(t)$ (stress relaxation) and an instantaneous stress $\dot{s}_{ik} = s_{ik}{}^{\circ}\delta(t)$ (creep):

Voigt model

$$s_{ik} = 2\mu_0 e_{ik} \left[1 + \frac{(t/\tau_\sigma)^{-\gamma}}{\Gamma(1-\gamma)}\right], \qquad e_{ik} = \frac{1}{2\mu_0} s_{ik} \Phi_{\gamma}\left(\frac{t}{\tau_\sigma}\right),$$

Maxwell model

$$s_{ik} = 2\mu_\infty e_{ik} \left[1 - \Phi_{\gamma}\left(\frac{t}{\tau_\varepsilon}\right)\right],$$

$$e_{ik} = \frac{1}{2\mu_\infty} s_{ik} \left[1 + \frac{(t/\tau_\varepsilon)^{\gamma}}{\Gamma(1+\gamma)}\right], \tag{7}$$

standard linear-body model

$$s_{ik} = 2e_{ik} \left[\mu_\infty - (\mu_\infty - \mu_0) \Phi_{\gamma}\left(\frac{t}{\tau_\varepsilon}\right)\right],$$

$$e_{ik} = \frac{1}{2} s_{ik} \left[\frac{1}{\mu_\infty} + \left(\frac{1}{\mu_0} - \frac{1}{\mu_\infty}\right) \Phi_{\gamma}\left(\frac{t}{\tau_\sigma}\right)\right],$$

$$\Phi_{\gamma}\left(\frac{t}{\tau}\right) \equiv \text{Э}_{\gamma}(t, \tau) * 1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{\gamma(n+1)}}{\Gamma[\gamma(n+1)+1]}. \tag{8}$$

When $\gamma = 1$ the Rabotnov function degenerates to the exponential function $\text{Э}_1(t, \tau) = \tau^{-1} e^{-t/\tau}$, and $\Phi_1(t/\tau) = 1 - e^{-t/\tau}$, and the corresponding equations describe an exponential variation of the functions. The creep of a Maxwell medium for $\gamma = 1$ is a linear function of time:

$$e_{ik} = \frac{1}{2\mu_\infty} s_{ik} \left[1 + \frac{t}{\tau_\varepsilon} \right]. \qquad (9)$$

For the Voigt model with $\gamma = 1$, stress relaxation will be absent, which follows from the relation

$$s_{ik} = 2\mu_0 \left[e_{ik} + e_{ik}^{\circ} \tau_\sigma \delta(t) \right], \qquad (10)$$

where we have used the representation

$$\delta(t) = \lim_{\gamma \to 1} \left[t^\gamma \Gamma(1-\gamma) \right]^{-1}. \qquad (11)$$

In the case of uniaxial compression or extension we can obtain the following expressions for the kernels of the integral operators for the Young moduli if we assume that the kernels of the operators for the shear modulus are described by Eq. (6), while volume deformations do not relax and are described by a constant bulk modulus K:

Voigt model

$$E(t) = E_\infty \delta(t) - (E_\infty - E_0)\, \vartheta_\gamma(t, \tau_\varepsilon), \qquad (\tau_\varepsilon < \tau_\sigma),$$

$$\frac{1}{E(t)} = \frac{1}{E_\infty}\delta(t) + \left(\frac{1}{E_0} - \frac{1}{E_\infty}\right) \vartheta_\gamma^-(t, \tau_\sigma),$$

$$E_\infty = 9K, \quad E_0 = 9K\mu_0 [3K + \mu_0]^{-1}, \quad (\tau_\varepsilon / \tau_\sigma)^\gamma = E_0 / E_\infty,$$

Maxwell model

$$E(t) = E_\infty [\delta(t) - \vartheta_\gamma(t, \tau_\sigma)], \qquad (E_\infty < 3\mu_\infty, \quad \tau_\varepsilon < \tau_\sigma),$$

$$\frac{1}{E(t)} = \frac{1}{E_\infty}\left[\delta(t) + \frac{1}{\tau_\sigma \Gamma(\gamma)} (t/\tau_\sigma)^{\gamma-1}\right],$$

$$E_\infty = 9K\mu_\infty [3K + \mu_\infty]^{-1}, \quad (\tau_\varepsilon / \tau_\sigma)^\gamma = E_\infty / 3\mu_\infty,$$

standard linear-body model

$$E(t) = E_\infty \delta(t) - (E_\infty - E_0)\, \vartheta_\gamma(t, \tau),$$

$$(E_\infty > E_0, \ \tau_\varepsilon < \tau < \tau_\sigma),$$

$$\frac{1}{E(t)} = \frac{1}{E_\infty}\delta(t) + \left(\frac{1}{E_0} - \frac{1}{E_\infty}\right) \vartheta_\gamma(t, \tau_\sigma),$$

$$E_\infty = \frac{9K\mu_\infty}{3K + \mu_\infty}, \quad E_0 = \frac{9K\mu_0}{3K + \mu_0},$$

$$\left(\frac{\tau}{\tau_\sigma}\right)^\gamma = \frac{E_0}{E_\infty}, \quad \left(\frac{\tau}{\tau_\varepsilon}\right)^\gamma = \frac{E_0 \mu_\infty}{E_\infty \mu_0}, \qquad (12)$$

where E_∞ and E_0 are the unrelaxed and relaxed values of the Young modulus.

Comparison of Eqs. (6) and (12) shows that if the shear deformation is described by the ϑ operators, the uniaxial compressions or extensions are also described by ϑ operators. In the Maxwell model there is a replacement of τ_ε by τ_σ, while for the standard linear-body the replacement $\tau_\varepsilon \to \tau$ occurs only for the direct operator E^* since the reciprocal operator $1/E^*$ is characterized by the relaxation time τ_σ, as before. We note that for the Voigt model the ϑ operator is described by the direct and the reciprocal Young modulus operators. It is only the operator $1/E^*$ for the Maxwell model that is expressed in terms of the fractional differentiation operator, as before.

Let us now discuss the dynamic characteristics of the above models under harmonic shear deformation. They can be deduced by introducing the replacement $p \to i\omega$ in the expressions given by Eq. (3). When this is done, and the real imaginary parts of the complex moduli of elasticity and compliance are separated, we obtain the parametric form of the equations relating the real and imaginary parts of the functions. The corresponding graphs for all the models are shown in the figure (a represents Voigt, b represents Maxwell, and c represents standard linear-body). Arrows indicate the direction in which $\omega\tau$ increases. It is clear from the figure that the fractional-exponential kernels correspond to circular arcs with a central angle $\pi\gamma$, while fractional derivatives correspond to straight lines at an angle $\pi\gamma/2$ to the real axis.

In spite of the fact that in all three models the elastic-aftereffect properties of a material can be described by the ϑ function (in the first case for creep, in the second for stress relaxation, and in the last for both cases), the range of validity of the models is quite different in each case.

The most general model is the standard linear body characterized by finite values of the relaxed and unrelaxed elasticity and compliance moduli. In the Voigt model $\mu_\infty \to \infty$, while in the Maxwell model $\mu_0^{-1} \to \infty$. Materials which show an unrestricted increase in the elastic modulus with increasing frequency do not appear to exist. Therefore, the modified Voigt model can be used only as a very rough approximation to estimate the low-frequency asymptotic behaviour of certain materials.

The modified Maxwell model assumes the existence of flow. The usual Maxwell model has been used for the qualitative description of the internal friction of metals [4, 5] and the relaxation of linear polymers in the transition region from the highly elastic to the viscous-flow state [6]. A disadvantage of the ordinary Maxwell model is that it involves a single relaxation time, while real solids have a broad relaxation spectrum. Therefore, if the existence of the relaxation spectrum in the region under consideration is important, it can be approximately taken into account by the modified Maxwell model. This approach was used to describe the dislocation background in the internal friction of metals [7, 8]. Apparently, it can also be used for the description of the elastico-viscous behavior of some polymers. It follows from Eq. (3) that the frequency dependence of the dynamic viscosity, $\mathrm{Re}\,\eta(\omega) \sim \omega^{-1-\gamma}$, occurs for strongly crystallized linear polyethylene with a density of 0.965 g/cm^3 in the region where the viscosity changes from 10^{12} to 10^{-2} poise [9].

The modified standard linear-body model can be used to describe relaxation phenomena in polymers in the region of the glassy state when relaxation is due to reorientation in the external force field of the side chains of molecules or finer kinetic units. The corresponding maxima in the mechanical loss factor $\mathrm{tg}\psi$ are well defined and usually have a symmetric bell-shaped form as a function of the logarithm of the frequency. It must be remembered, however, that they are accompanied by a relatively high background, $\mathrm{tg}\psi \sim 10^{-3}$, which is almost independent of frequency [9].

The most interesting region from the point of view of the elastic-aftereffect is the α-relaxation region of polymers, which is connected with the conformal motion of the main chain of molecules, and is characterized by a relatively broad relaxation spectrum. However, the distribution function for the logarithms of the relaxation times is essentially asymmetric in this region, and approaches a straight line with a slope of $-1/2$. Since as soon as a maximum is reached the distribution function does not tend to fall to a level characterized by large τ, it may not be adequately approximated by a symmetric distribution function such as the ϑ function [10]. Therefore, the description of the elastic aftereffect in polymers in this region in terms of the fractional exponential functions may frequently be only qualitatively correct as far as rheological properties are concerned. For metals, in which the relaxation peaks corresponding to different mechanisms are well separated, the use of the modified standard linear-body model is restricted only by the requirement that the system be linear.

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